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Weighted Polynomial Approximation

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DEDICATED TO THE MEMORY OF GÉZA FREUD

The subject of weighted polynomial approximation on the whole real line was the main interest of Professor Freud during the last ten years or so of his life. In this paper we shall survey some aspects of the final stages of his efforts. We do not intend to be exhaustive, however, and thus if we fail to mention certain results here, it does not imply that we consider them uninteresting.

As with many other problems in approximation theory, the subject of weighted approximation goes back to S. N. Bernstein who posed the following problem: Let w be a continuous function on $\mathbb{R}$ such that $x^n w(x) \to 0$ as $|x| \to \infty$, $n = 0, 1, 2, \ldots$. Find necessary and sufficient conditions on w for $\{x^n w(x)\}$ to be fundamental in $C_0(\mathbb{R})$. While this problem was investigated in great detail by several mathematicians, (see, e.g., [1, 34, 58] for reviews), the important question of determining the degree of approximation by "weighted polynomials" (i.e., expressions of the form $w(x)\,P(x)$, where P is a polynomial) was rarely discussed. A notable exception was the work of M. M. Dzrbasyan and his collaborators. Out of historical interest, we note one of their results.

THEOREM 1. [12] *Let* $w(x) := \exp(-Q(x))$ *where* Q *is even, nonnegative, differentiable on* $(0, \infty)$. *Further, let* $Q(x)/x \uparrow \infty$ *as* $x \to \infty$, $Q(p_n) = n$, $n = 1, 2, \ldots$, *and* f *be a function which is bounded along with its first* s *derivatives on* $\mathbb{R}$. *Then, for* $n = 1, 2, \ldots$, *there is a polynomial* P_n *of degree at most* n *such that*

$$\max_{x \in \mathbb{R}} |w(x)[f(x) - P_n(x)]| \leqslant c \left(\frac{p_n}{n}\right)^s \sup_{y \in \mathbb{R}} |f^{(s)}(y)|, \tag{1}$$

where c *is a constant depending only on* Q *and* s.

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Dzrbasyan also obtained a converse theorem [10, 11] which, however, does not "match" the above direct theorem to yield an equivalence theorem. Nevertheless, Byrnes and Newman [6] proved that in the case of the weights $\exp(-|x|^{\beta})$, where $\beta > 1$, the Theorem 1 is unimprovable. A major drawback in these investigations was that one had to assume that f, or its derivatives are themselves bounded on $\mathbb{R}$. Writing $g := wf$, it is to be expected that the degree of approximation of g would depend upon the growth of g near $\pm\infty$ and not just on its smoothness. However, the techniques used in the above investigations required the unduly (on hindsight) restrictive condition that gw^{-1} be bounded.

One of the most important contributions which Professor Freud made to the study of weighted approximation was the development of both the direct and converse theorems on weighted approximation (which "match") for a very general class of weight functions and with only a minimal assumption on the function being approximated. He worked very hard on this problem, "making progress inch-by-inch" to quote him; despite a multitude of personal problems. A lot of beautiful papers came out of this effort and especially the theory of orthogonal polynomials was considerably enriched in the process. We now proceed to state the direct and converse theorems as they stand today. We shall mention only a special case, which in our opinion, is the most completely studied. For the exact assumptions under which various statements are true, the interested reader may consult [24].

We shall assume that $w(x) = w_Q(x) = \exp(-Q(x))$, where Q satisfies the following conditions

(1) Q is an even function in $C^2(0, \infty)$.

(2) Q'' is positive and nondecreasing on $(0, \infty)$.

(3) There are constants c and d such that

$$1 \leqslant c \leqslant x \frac{Q''(x)}{Q'(x)} \leqslant d < \infty \qquad (x > 0). \tag{2}$$

Examples of such weight functions include $\exp(-|x|^{\alpha})$, $\alpha \geqslant 2$.

Next, let us introduce the expressions which will serve as the moduli of continuity and smoothness. Let $\delta > 0$, $1 \leqslant p \leqslant \infty$ and $w_Q f \in L^p(\mathbb{R})$.

Put

$$Q'_\delta(x) := \min\{\delta^{-1}, (1 + Q'(x)^2)^{1/2}\}, \tag{3}$$

$$\omega_1(p, Q, f, \delta) := \sup_{|t| \leqslant \delta} \|\Delta_t(w_Q f)\|_p + \delta\, \|Q'_\delta w_Q f\|_p, \tag{4}$$

$$\omega_2(p, Q, f, \delta) := \sup_{|t| \leqslant \delta} \|\Delta_t^2(w_Q f)\|_p + \delta \sup_{|t| \leqslant \delta} \|Q'_\delta \Delta_t(w_Q f)\|_p + \delta^2\, \|Q'^2_\delta w_Q f\|_p, \tag{5}$$

where

$$\Delta_t g(x) := g(x+t) - g(x), \Delta_t^2 g := \Delta_t(\Delta_t g). \tag{6}$$

The expressios which were found suitable as the first and second order moduli of continuity are then defined below in (7), (8), respectively.

$$\Omega_1(p, Q, f, \delta) := \inf_{a \in \mathbb{R}} \omega_1(p, Q, f-a, \delta), \tag{7}$$

$$\Omega_2(p, Q, f, \delta) := \inf_{a,b \in \mathbb{R}} \omega_2(p, Q, f-a-bx, \delta). \tag{8}$$

While we shall elaborate more on these later, let us only note here that the "additional terms" in the definitions of ω_1 and ω_2 serve to measure the growth of $w_Q f$ and its differences near $\pm\infty$; while the inf in (7) and (8) are needed to preserve the obvious elementary properties whicn one expects from a modulus of continuity.

Next, we introduce the degree of approximation. Let π_n denote the class of all polynomials of degree at most n. Set

$$\varepsilon_n(p, Q, f) := \inf_{P \in \pi_n} \|w_Q(f-P)\|_p. \tag{9}$$

Finaly, let q_n be the least positive number such that

$$q_n Q'(q_n) = n. \tag{10}$$

The direct and converse theorems can now be stated as follows:

THEOREM 2. [24] *Let $s \geq 0$, $n \geq 2s$ be natural numbers, $1 \leq p \leq \infty$, $w_Q f \in L^p(\mathbb{R})$.*

(a) *If f is s-times differentiable (i.e., f is an s-times iterated integral of a locally integrable function $f^{(s)}$) and $w_Q f^{(s)} \in L^p(\mathbb{R})$ then, for $r = 1, 2$,*

$$\varepsilon_n(p, Q, f) \leq c\left(\frac{q_n}{n}\right)^s \Omega_r\left(p, Q, f^{(s)}, \frac{q_n}{n}\right). \tag{11}$$

(b) *If*

$$\sum_{k=1}^{\infty} k^{s-1} q_k^{-s} \varepsilon_k(p, Q, f) < \infty \tag{12}$$

then f is s-times differentiable, $w_Q f^{(s)} \in L^p(\mathbb{R})$, and for $r = 1, 2$,

$$\Omega_r\left(p, Q, f^{(s)}, \frac{q_n}{n}\right) \leq c\left\{\left(\frac{q_n}{n}\right)^r \sum_{k=0}^{n} k^{r+s-1} q_k^{r-s} \varepsilon_k(p, Q, f) + \sum_{k=[n/2]}^{\infty} k^{s-1} q_k^{-s} \varepsilon_k(p, Q, f)\right\}. \tag{13}$$

(c) *For* $r = 1, 2$,

$$\Omega_r\left(p, Q, f, \frac{q_n}{n}\right) \leqslant c\left\{\left(\frac{q_n}{n}\right)^r \sum_{k=0}^{n} (k+1)^{r-1} q_{k+1}^r \varepsilon_k(p, Q, f)\right\}. \tag{14}$$

In this theorem as well as in others to follow, c, c_1, c_2 would always denote constants independent of the obvious variables, not necessarily having the same value even if they occur twice in the same formula.

We note the following equivalence theorem which follows from Theorem 2:

THEOREM 3. *Suppose* $r = 1, 2$, $0 < \alpha < r$, $s \geqslant 0$, *is an integer*, $\rho = s + \alpha$, $1 \leqslant p \leqslant \infty$ *and* $w_Q f \in L^p(\mathbb{R})$. *Then the following are equivalent*

(a) $\Omega_r(p, Q, f^{(s)}, \delta) = \mathcal{O}(\delta^\alpha)$.

(b) $\varepsilon_n(p, Q, f, \delta) = \mathcal{O}((q_n/n)^\rho)$.

While the theorems are certainly very interesting in their own right; in our opinion, the ideas which Professor Freud developed and which eventually led to these theorems are even more interesting.

One obvious novelty is the introduction of the modified moduli of continuity. It turns out that they are equivalent to certain K-functionals. For an integer $r \geqslant 1$, $\delta > 0$ and $w_Q f \in L^p(\mathbb{R})$, where $1 \leqslant p \leqslant \infty$, let

$$K_{(r)}(p, Q, f, \delta) := \inf\{\|w_Q(f-g)\|_p + \delta^r \|g^{(r)} w_Q\|_p\}, \tag{15}$$

where the inf is over all r-times differentiable g such that $w_Q g^{(r)} \in L^p(\mathbb{R})$. Then [13, 25], for $r = 1, 2$,

$$K_{(r)}(p, Q, f, \delta) \sim \Omega_r(p, Q, f, \delta), \tag{16}$$

where, as usual, by $A \sim B$ we mean that $cA \leqslant B \leqslant c_1 A$. As far as we know, this was the first instance when approximation theoretic results were conscientiously obtained first in terms of the K-functionals and then these computed to yield the moduli of continuity. To underline this, we note that while the analogues of Theorems 2 and 3 are known with higher order K-functionals [24], no explicit expressions for these high order K-functionals are known in general (see however, [14]). The idea of using a modified modulus of continuity for special purposes is no longer novel, however. In a way, Dzrbasyan himself used a slightly modified modulus of continuity in his work on weighted approximation [11]. In investigating how the smoothness of a function affects the smoothness of its Fourier transform; we used a modified K-functional and modulus of smoothness analogous to Ω_2 [39]. Freud's ideas directly inspired Ky's works [28, 29] on approximation of functions on $[-1, 1]$ with Jacobi weights. K-functionals with somewhat more general weights than the Jacobi weights were evaluated by Z. Ditzian [9] while yet another modulus of smoothness was studied by Ivanov [27] and Popov [53] in connection with algebraic polynomial approximation in $L^p[-1, 1]$; although of course, we are not really in a position to assess the extent to which Freud's ideas inspired the last three papers.

In addition to the evaluation of K-functionals, one needs analogues of the well-known Jackson–Favard estimates, Bernstein inequality and a sequence of linear operators providing an almost optimal approximation [8, 5]. The development of these was perhaps the most difficult part of Professor Freud's investigations. The novelty in this work was the use of estimates on orthogonal polynomials and related quantities.

Let for $n = 0, 1, \ldots,$

$$\begin{aligned} p_n(x) &:= \gamma_n x^n + \cdots = \gamma_n \prod_{k=1}^{n} (x - x_{kn}) \in \pi_n, \\ \gamma_n &> 0, \qquad x_{nn} < x_{n-1,n} < \cdots < x_{1n} = X_n, \end{aligned} \tag{17}$$

be the system of orthonormal polynomials with respect to w_Q^2:

$$\int p_n(x)\, p_m(x)\, w_Q^2(x)\, dx = \delta_{nm}. \tag{18}$$

In a series of papers, [15–20] Freud proved the following estimates:

$$\frac{1}{2} q_n \leqslant \frac{\gamma_{n-1}}{\gamma_n} \leqslant 2q_n, \tag{19}$$

$$\frac{1}{2} q_{n-1} \leqslant \max_{1 \leqslant k \leqslant n-1} \frac{\gamma_{n-1}}{\gamma_k} \leqslant X_n \leqslant 2 \cos \frac{\pi}{n+1} \max_{1 \leqslant k \leqslant n-1} \frac{\gamma_{k-1}}{\gamma_k} \leqslant 4q_{n-1}, \tag{20}$$

$$w_Q^2(x) \sum_{k=0}^{n-1} p_k^2(w_Q^2, x) \leqslant c \frac{n}{q_n}, \qquad x \in \mathbb{R}, \tag{21}$$

$$w_Q^2(x) \sum_{k=0}^{n-1} [p_k'(w_Q^2, x)]^2 \leqslant c \left(\frac{n}{q_n}\right)^3 \qquad x \in \mathbb{R}, \tag{22}$$

$$\int_{-\infty}^{\infty} W_Q^2(x)\, P(x)^2\, dx \leqslant (1 + e^{-cn}) \int_{-cq_n}^{cq_n} w_Q^2(x)\, P(x)^2\, dx, \quad P \in \pi_n. \tag{23}$$

Each of these inequalities has inspired a good deal of further research in the theory of orthogonal polynomials, cf. P. Nevai's review [47]. Using these estimates, certain techniques originally proposed by Carleman [7], and some other standard techniques in analysis such as the use of the duality principle [55], Riesz–Thorin interpolation theorem, etc., Freud was able to obtain the following results which proved to be central for the theory of weighted approximation.

Let $w_Q f \in L^p(\mathbb{R})$,

$$a_k := \int w_Q^2 f p_k \, dx, \tag{24}$$

$$s_m(f, x) := \sum_{k=0}^{m-1} a_k \, p_k(x), \tag{25}$$

$$v_n(f, x) = \frac{1}{2n} \sum_{m=n+1}^{2n} s_m(f, x), \tag{26}$$

THEOREM 4. [17, 16, 21]. (a) *Let* $w_Q f \in L^p(\mathbb{R})$. *Then*

$$\|w_Q v_n(f)\|_p \leqslant c \, \|w_Q f\|_p. \tag{27}$$

Hence, the linear operators v_n *satisfy*

$$\|w_Q(f - v_n(f))\|_p \leqslant c \cdot \varepsilon_n(p, Q, f), \tag{28}$$

$$v_n(f) \in \pi_{2n-1}, \quad v_n(P) = P \qquad \text{if} \quad P \in \pi_n \tag{29}$$

(b) (*Jackson–Favard estimates*) *Let* f *be differentiable,* $w_Q f' \in L^p(\mathbb{R})$. *Then*

$$\varepsilon_n(p, Q, f) \leqslant c \frac{q_n}{n} \varepsilon_{n-1}(p, Q, f'). \tag{30}$$

(c) (*Bernstein inequality*) *Let* $P \in \pi_n$. *Then*

$$\|w_Q P'\|_p \leqslant c \frac{n}{q_n} \|w_Q P\|_p. \tag{31}$$

As it is to be expected, such a major investigation did have quite a few side rewards. We state three theorems. Theorem 5 on the summability of orthonormal polynomial expansions was perhaps the starting point for the general study of weighted approximation. The theorem 6 on the analogue of the Dirichlet–Jordan criterion is directly related to the Jackson–Favard-type estimates; something which we find interesting because the usual proofs for the trigonometric case are quite different. Theorem 7 on Lagrange interpolation involves more or less everything known about weighted approximation until the demise of Professor Freud. We note, however, that what we state is only a special case.

THEOREM 5 [22]. *Let $w_Q f \in L^2$ and $x \in \mathbb{R}$ be an L^2-Lebesgue point of f, i.e.,*

$$|h|^{-1} \left| \int_x^{x+h} [f(t)-f(x)]^2 w_Q^2 \, dt \right| \to 0 \qquad as \quad |h| \to 0. \tag{32}$$

Then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} |s_m(f, x) - f(x)| = 0. \tag{33}$$

In particular, (33) *holds almost everywhere.*

THEOREM 6 [23]. *Let f be a continuous function on $\mathbb{R}$ which is of bounded variation over every finite segment.*

Moreover, let

$$\int w_Q \, |df| < \infty. \tag{34}$$

Then

$$\lim_{n \to \infty} |w_Q(x)[f(x) - s_n(f, x)]| = 0 \tag{35}$$

uniformly in $x \in \mathbb{R}$.

Remark. We proved recently [41], that the conditions on f in Theorem 6 are equivalent to the conditions that f be continuous and

$$\Omega_1(1, Q, f, \delta) = \mathcal{O}(\delta). \tag{36}$$

THEOREM 7 [3]. *In addition to the conditions on Q already stated, let* $\exp(Q(x))$ *be an entire function with nonnegative coefficients;*

$$\exp(Q(x)) = \sum_{n=0}^{\infty} A_n x^{2n}, \qquad A_n \geq 0, \quad n = 0, 1, 2, \ldots. \tag{37}$$

Suppose f is an absolutely continuous function on $\mathbb{R}$ such that

$$\int w_Q(t) \, t^n f(t) \, dt < \infty, \qquad n = 0, 1, 2, \ldots. \tag{38}$$

Moreover, let

$$\sup_{|t| \geq x} |tQ'(t)|^{1/2} w_Q(t) \, |f(t)| \to 0 \qquad as \quad x \to \infty. \tag{39}$$

Then

$$\sup_{x \in \mathbb{R}} w_Q(x)\,|s_n(f, x) - L_n(f, x)| \to 0 \qquad as \quad n \to \infty, \tag{40}$$

where $L_n(f) \in \pi_{n-1}$ *is the Lagrange interpolation polynomial for* f *at the zeros of* p_n:

$$L_n(f, x_{kn}) = f(x_{kn}), \qquad k = 1, 2, ..., n. \tag{41}$$

Needless to say, Freud's ideas have a considerable influence on the later work by his students such as Nevai [26, 32, 33, 48–52], Mhaskar [35–46], Bonan [4], and Al-Jarrah [2], as well as others such as Ullman [56, 57]. In recent years, the interest in weighted approximation and related questions is rapidly increasing. Thus, for example, Lubinsky [30] studies geometric convergence of quadrature formulae, Zalik [59–61] studies analogues and extensions of the Bernstein inequality and the inequality (23), as well as weighted Müntz-type approximation, Rahmanov [54] studies the convergence of orthogonal polynomial series in the complex domain. Moreover, questions concerning the best constants are also being studied. For example, using (21) and (23), we obtained in [35] the following Nikolskii-type inequalities:

THEOREM 8. *Let* $1 \leqslant p, r \leqslant \infty$, $P \in \pi_n$. *Then*

$$\|w_Q P\|_r \leqslant c \left(\frac{n}{q_n}\right)^{|(1/p) - (1/r)|} \|w_Q P\|_p. \tag{42}$$

Markett [31] has recently proved (although apparently unaware of our work) that in the case of the Hermite weight, $w_Q(x) = \exp(-x^2/2)$, the exponent $|(1/p) - (1/r)|$ is the best possible.

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I would like to thank Paul Nevai for inviting me to pay this homage to my beloved teacher.

Note added in proof. A great deal of progress has been made in the theory of weighted approximation since this survey was written. In particular, Professor V. Totik has kindly informed me the following:

(1) The K-functionals of all orders have been computed.

(2) Bernstein-type inequalities as well as Favard-type estimates are now known for more general weights, including $\exp(-|x|^\alpha)$, $\alpha > 0$.

(3) Nikolskii-type inequalities have been proved to be exact, at least for $\exp(-|x|^\alpha)$, $\alpha > 0$.

In addition, Nikolskii-type inequalities have been extended to other weights and exponents; and a fairly detailed asymptotics for orthogonal polynomials with respect to $\exp(-x^4)$ and $\exp(-x^6)$ is known. A somewhat technical, but potentially very important progress is that every very strongly regular weight function is equivalent to another which also satisfies the conditions of Theorem 7.

Among the principal contributors are S. Bonan, Z. Ditzian, D. S. Lubinsky, A. Maté, H. N. Mhaskar, P. Nevai, E. B. Saff, V. Totik, J. L. Ullman and R. A. Zalik.

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